J. Appl. Maths Mechs, Vol. 55, No. 3, pp. 376-384, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00
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THE MOTION OF A CARRIAGE WITH CONSTANT VELOCITY ALONG A BEAM OF INFINITE LENGTH RESTING ON A BASE WITH TWO ELASTIC CHARACTERISTICS*

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The vertical oscillations of an infinite Bernoulli-Euler beam resting on a viscous inertial base with two elastic characteristics under the action of a deforming carriage which is continuously moving at a constant velocity is considered. The carriage, consisting of a system of rigid bodies with viscoelastic couplings between them, makes contact with the beam via viscoelastic springs at a finite number of points which allows the small vertical oscillations of the elements of the carriage to be described by a system of ordinary differential equations with constant coefficients. A technique is proposed for obtaining the asymptotic forms of the solution at long times. The asymptotic forms of the solutions are presented and they are analysed for certain types of loads.

Unlike similar problems which have been considered previously (a review of the literature is given in /1/), the formulae obtained enable one to construct asymptotic solutions which describe the interaction of the deforming carriage with the beam and, in particular, to investigate the rate and nature of the propagation of perturbations in the beam, the amplitude - frequency characteristics of the steady-state oscillatory regimes and the rate of convergence of a non-stationary solution to them and the growth in the flexure of the beam accompanying motions at critical velocities and when an oscillatory force at a resonant frequency acts on the carriage.

1. Formulation of the problem. The reaction of the base is described by the differential equation

$$r_*(x_*, t_*) = -m_0 \frac{\partial^2 w_*}{\partial t_*^2} + k_{1*} \frac{\partial^2 w_*}{\partial x_*^2} - \lambda_* \frac{\partial w_*}{\partial t_*} - k w_*$$

where $w_* = w_*(x_*, t_*)$ is the flexure of the base, m_0, k_{1*}, k and λ_* are the virtual mass per metre, the elastic characteristics and the viscosity of the base respectively, t_* is the current time and x_* is the coordinate of a section.

We shall use the following notation: EJ is the rigidity of the beam, ρ is its mass per metre, v_* is the velocity of motion of the carriage, $\xi_* = x_* - v_* t_*$ is the coordinate of a cross-section of the beam, $y_* = y_*(\xi_*, t_*)$ is the deflection of the beam, $z_{n*}(t_*)$ is the displacement in the n-th spring, ξ_{n*} is its coordinate, $y_{n*} = y_*(\xi_{n*}, t_*)$ is the deflection of the beam under the spring, m_{n*} are concentrated masses at the points ξ_{n*}, $P_{n*}(t_*)$ are the forces applied to them, e_{n*} and μ_{n*} are the coefficients of elasticity and viscosity of the n-th spring and $q_{n*}(t_*)$ is the pressure of the carriage on the beam at a point ξ_{n*} (Fig.1).

We now introduce the basic similarity coefficients

$$R_1 = \left(\frac{EJ}{k}\right)^{1/4}, \qquad R_2 = \left(\frac{2k}{\rho + m_0}\right)^{-1/2} \qquad R_3 = (EJk)^{1/2}$$

with the dimensions of m, s and H, respectively, and write the initial equations in dimensionless variables in a travelling coordinate system associated with the carriage. The transition to dimensional quantities, which are denoted by an asterisk, is made by multiplying the dimensionless variables by the combinations of basic similarity coefficients which correspond to their dimensions.

The equation for the bending of the beam is

$$\frac{\partial^4 y}{\partial \xi^4} + (2v^2 - k_1)\frac{\partial^2 y}{\partial \xi^2} - 4v\frac{\partial^2 y}{\partial \xi \partial t} + 2\frac{\partial^2 y}{\partial t^2} - \lambda v \frac{\partial y}{\partial \xi} + \lambda \frac{\partial y}{\partial t} + y = \sum_{n=1}^{N} q_n(t)\,\delta(\xi - \xi_n) \qquad (1.1)$$

*Prikl.Matem.Mekhan.,55,3,461-471,1991

and the conditions for the carriage to be in contact with the beam $(n = 1, 2, \ldots, N)$ are

$$m_n \frac{d^2 y_n}{dt^2} + \mu_n \left(\frac{dy_n}{dt} - \frac{dz_n}{dt} \right) + \varepsilon_n (y_n - z_n) = P_n(t) - q_n(t) \tag{1.2}$$

Small vertical oscillations of the elements of the carriage are described by linear ordinary differential equations with constant coefficients, while the relationship between the deformation of the springs and the displacements of the elements of the carriage is described by linear algebraic equations. The boundary and initial conditions are assumed to be null conditions.

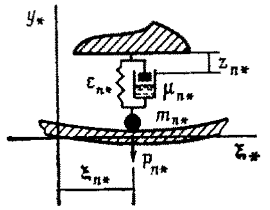

Fig.1

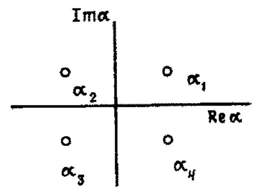

Fig.2

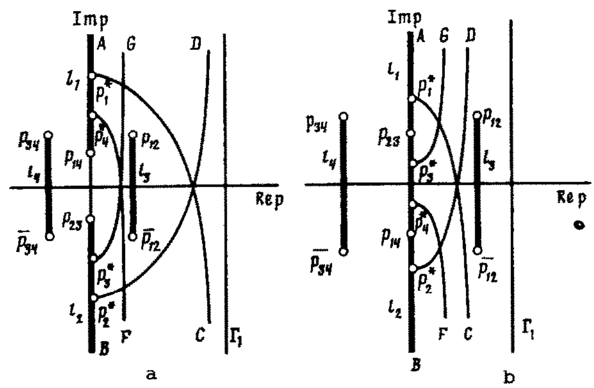

Fig.3

2. ***A technique for constructing the asymptotic solution when*** $t \to \infty$. Let us successively apply a Fourier transformation with respect to ξ and a Laplace transformation with respect to t to Eq.(1.1). Then,

$$y(\xi, t) = \frac{\exp(-{}^1/_4\lambda t)}{4\pi^2 i} \int_{\Gamma_1} \int_{\Gamma_2} Y_{LF}(p, \alpha) \exp(pt - i\alpha\xi)\, d\alpha\, dp \tag{2.1}$$

$$Y_{LF}(p, \alpha) = \frac{1}{\Phi(\alpha, p)} \sum_{n=1}^{N} Q_n(p) \exp(i\alpha\xi_n) \tag{2.2}$$

$$\Phi(\alpha, p) = \alpha^4 - (2v^2 - k_1)\alpha^2 + 4ivp\alpha + 2p^2 + \gamma$$

$$\gamma = 1 - {}^1/_8\lambda^2 \tag{2.3}$$

$$Q_n(p) = \int_0^\infty q_n(t) \exp[({}^1/_4\lambda - p)t]\, dt \tag{2.4}$$

where Γ_1 is a straight line lying to the right of the $\operatorname{Re} p = {}^1/_4\lambda$ axis in the p-plane and Γ_2 is the real axis in the α-plane.

Table 1. $0 \leqslant v \leqslant v_0$

$\operatorname{Im}\alpha_1 \geqslant 0,\ \operatorname{Im}\alpha_4 \leqslant 0;$ $\operatorname{Im}\alpha_2 = \operatorname{Im}\alpha_3 = 0,\ \operatorname{Re}\alpha_3 \leqslant \operatorname{Re}\alpha_2$	$p \in l_2^+$
$\operatorname{Im}\alpha_2 \geqslant 0,\ \operatorname{Im}\alpha_3 \leqslant 0;$ $\operatorname{Im}\alpha_1 = \operatorname{Im}\alpha_4 = 0,\ \operatorname{Re}\alpha_1 \leqslant \operatorname{Re}\alpha_4$	$p \in l_1^+$
$\operatorname{Im}\alpha_1 > 0,\ \operatorname{Im}\alpha_2 > 0,\ \operatorname{Im}\alpha_3 < 0,\ \operatorname{Im}\alpha_4 < 0$	$p \notin l_1 \cup l_2$

Table 2. $v > v_0$

$\mathrm{Im}\,\alpha_1 > 0,\ \ \mathrm{Im}\,\alpha_4 < 0;$ $\mathrm{Im}\,\alpha_2 = \mathrm{Im}\,\alpha_3 = 0,\ \ \mathrm{Re}\,\alpha_3 < \mathrm{Re}\,\alpha_2$	$p \in l_2^+ \setminus l_1^+$
$\mathrm{Im}\,\alpha_k = 0,\ k = 1, 2, 3, 4,$ $\mathrm{Re}\,\alpha_3 \leqslant \mathrm{Re}\,\alpha_2 < \mathrm{Re}\,\alpha_1 \leqslant \mathrm{Re}\,\alpha_4$	$p \in l_1^+ \cap l_2^+$
$\mathrm{Im}\,\alpha_2 > 0,\ \ \mathrm{Im}\,\alpha_3 < 0;$ $\mathrm{Im}\,\alpha_1 = \mathrm{Im}\,\alpha_4 = 0,\ \ \mathrm{Re}\,\alpha_1 < \mathrm{Re}\,\alpha_4$	$p \in l_1^+ \setminus l_2^+$
$\mathrm{Im}\,\alpha_1 > 0,\ \mathrm{Im}\,\alpha_2 > 0,\ \mathrm{Im}\,\alpha_3 < 0,\ \mathrm{Im}\,\alpha_4 < 0$	$p \notin l_1 \cup l_2$

Let us now consider a function of the roots of the polynomial (2.3), $\alpha(p)$. Its branching points are determined from the equation

$$R(\Phi, \Phi_\alpha') = 0$$

where $R(\Phi, \Phi_\alpha')$ is the resultant of the polynomial Φ and Φ_α' /2/. From this, we obtain

$$\begin{gathered} 8p^6 + (12\gamma + v^4 - 10k_1v^2 - 2k_1^2)\,p^4 + \\ [6\gamma^2 - \gamma\,(k_1 - 2v^2)\,(2k_1 + 5v^2) + {}^1/_8 k_1\,(k_1 - 2v^2)^3]p^2 + \\ \gamma\,[\gamma - {}^1/_4\,(k_1 - 2v^2)^2]^2 = 0 \end{gathered} \tag{2.5}$$

We shall denote the branching points p_{ij} in accordance with the splicing of the branches of the function $\alpha(p)$ which is realized at these points. Eq.(2.5) indicates that there are two third-order branching points when $v = 0, k_1 = 0$ and six first-order branching points when $v > 0$ or $k_1 > 0$. By considering an analytic continuation from the real axis (the location of the roots α_k of the polynomial (2.3) in the α-plane is shown for this case in Fig.2), it is possible to construct a domain where the function $\alpha(p)$ is single-valued and the cuts then pass along the rays $l_1 = [p_{14}, +i\infty)$, $l_2 = [p_{23}, -i\infty)$ and the simple curves l_3 and l_4 joining the points $p_{12}, \bar{p}_{12}$ and $p_{34}, \bar{p}_{34}$ (The right and left banks of the cuts are subsequently indicated by a plus and a minus sign, respectively). The location of the cuts l_k in the p-plane is shown in Figs.3,a and b, respectively, for the cases when $0 < v < v_0$ and $v > v_0$ (the limiting position of the cuts l_1 and l_2 and the domain of sinlge-valuedness where the function $\alpha(p)$ is to be taken as being single-valued are shown in Fig.3,b). The conditions for the numbering of the roots of the polynomial (2.3), $\alpha_k(p)$ are presented in Tables 1 and 2. Here,

$$v_0 = \sqrt{\pm\sqrt{\gamma} + k_1/2} \tag{2.6}$$

which corresponds to the coincidence of the branching points p_{14} and p_{23} on the real axis. In practice, however, k_1 is small and we shall therefore take the plus sign in the equality (2.6).

We note that the non-rigorous inequalities presented in Tables 1 and 2 can only turn into equalities at the branching points p_{14} and p_{23}. What is more, in order to obtain the values of the branches of the function $\alpha(p)$ on l_1^- and l_2^-, it is necessary to take account of the obvious permutation $\alpha_i \leftrightarrow \alpha_j$ on passing around the branching points p_{ij}.

Using the residues, let us now find the Laplace transform of the function (2.1)

$$Y_L(p, \xi) = \sum_{n=1}^{N} Q_n(p)\,h_n(p, \xi) \tag{2.7}$$

$$h_n(p, \xi) = i \sum_{k=1}^{2} \frac{\exp[-i\alpha_k(p)(\xi - \xi_n)]}{\Phi_\alpha'(\alpha_k, p)} \qquad (\xi \leqslant \xi_n)$$

$$h_n(p, \xi) = -i \sum_{k=3}^{4} \frac{\exp[-i\alpha_k(p)(\xi - \xi_n)]}{\Phi_\alpha'(\alpha_k, p)} \qquad (\xi \geqslant \xi_n)$$

In the case of the functions $h_n(p, \xi)$, only the branching points p_{14} and p_{23} are singular since the functions $h_n(p, \xi)$ are symmetric with respect to the permutations $\alpha_1 \leftrightarrow \alpha_2$, $\alpha_3 \leftrightarrow \alpha_4$ and they have a finite limit at the points $p_{12}, \bar{p}_{12}, p_{34}, \bar{p}_{34}$. For the same reason, the functions $h_n(p, \xi)$ do not tolerate discontinuities in passing over l_3 and l_4.

Let us put $\xi = \beta t + \eta$, where β is the group velocity of a packet of waves with a wave number α and write function (2.1), allowing for relationships (2.7), in the form

$$y(\eta, t) = \sum_{n=1}^{N} y_n(\eta, t) \tag{2.8}$$

$$y_n(\eta, t) = I_{n1}(\eta, t) + I_{n2}(\eta, t)$$
$$\text{for } \beta < 0 \text{ or } \beta = 0,\ \eta \leqslant \xi_n$$
$$y_n(\eta, t) = -I_{n3}(\eta, t) - I_{n4}(\eta, t)$$
$$\text{for } \beta > 0 \text{ or } \beta = 0,\ \eta \geqslant \xi_n$$
$$I_{nk}(\eta, t) = \frac{\exp(-{}^1/_4\lambda t)}{2\pi i} \int_{\Gamma_1} f_{nk}(p) \exp[tS_k(p, \alpha)]\, dp$$
$$f_{nk}(p) = \frac{iQ_n(p) \exp[-i\alpha_k(p)(\eta - \xi_n)]}{\Phi_\alpha'(\alpha_k, p)}$$
$$S_k(p, \beta) = p - i\alpha_k(p)\beta$$

Let us now find the asymptotic forms of the integrals I_{nk} for large t by the method of steepest descent. We shall consider the case when the poles and the branching points p_{14}, p_{23} are the singular points of the functions $Q_n(p)$. (This condition is satisfied a fortiori if pulsed, constant or oscillatory forces are applied to the elements of the carriage).

The crossing points of the functions S_k $(\beta \neq 0)$ are determined from the equation

$$R(\Phi, \Psi) = 0,\ \Psi(\alpha, p) = \Phi_\alpha'(\alpha, p) + i\beta\Phi_p'(\alpha, p)$$

and, from this, we obtain

$$\begin{aligned} &8p^6 + [12\gamma + (\beta^2 - v^2)^2 + 2(\beta^2 - 5v^2)k_1 - 2k_1^2]\, p^4 + \\ &\{2\gamma[3\gamma + v(v + \beta)(5v^2 - 5\beta v + 2\beta^2)] - (v^2 - \beta^2)[\gamma + \\ &v^2(v^2 - \beta^2)]\, k_1 - {}^1/_2[4\gamma - v(3v^3 - \beta^2 v - 2\beta^3)]\, k_1^2 - \\ &-{}^1/_4(3v^2 + \beta^2)\, k_1^3 + {}^1/_8 k_1^4\}\, p^2 + \\ &\gamma\{[\gamma - v^2(v^2 - \beta^2)]^2 + 2v^2(\gamma + \beta^2 v^2 - v^4)\, k_1 - \\ &{}^1/_2(\gamma + \beta^2 v^2 - 3v^4)\, k_1^2 - {}^1/_2 v^2 k_1^3 + {}^1/_{16} k_1^4\} = 0 \end{aligned} \tag{2.9}$$

Since the group velocity β depends on the real number α, it is necessary solely to consider those roots of the polynomial which are located on l_1 and l_2. These roots determine the third-order crossing points when $v = -\beta$, $k_1 = 0$ and the first-order crossing points when $v \neq -\beta$ or $k_1 > 0$. We shall denote them by p_k^* for the corresponding functions S_k. It is noted that the points p_1^*, p_2^* are defined when $\beta < 0$, while p_3^*, p_4^* are defined when $\beta > 0$, and $p_1^* \in l_1^+$, $p_4^* \in l_1^+$, $p_2^* \in l_2^+$, $p_3^* \in l_2^+$.

The functions $h_n(p, \xi)$ do not have discontinuities on l_3 and l_4. Hence, the sum of the integrals $I_{n1} + I_{n2}, I_{n3} + I_{n4}$ along the edges of these cuts are equal to zero and the initial contour Γ_1 can be deformed when $\beta \neq 0$ into the contours Γ_{1k} which satisfy the equation

$$\operatorname{Re} S_k(p, \beta) = 0,\ (k = 1, 2, 3, 4)$$

The pass contours $\Gamma_{11} = Ap_1^*C$, $\Gamma_{12} = Bp_2^*D$, $\Gamma_{13} = Bp_3^*G$, $\Gamma_{14} = Ap_4^*F$ $(A = +i\infty,\ B = -i\infty,\ A \in l_1^+,\ B \in l_2^+)$ are shown in Fig.3a and b for certain values of β.

The contours Γ_{1k} possess the following properties:

a) just a single contour Γ_{1k} passes through any point of the half plane $\operatorname{Re} p > 0$ for a certain β and a fixed k;

b) the contours Γ_{11} and Γ_{12}, Γ_{13} and Γ_{14} are pairwise symmetrical about the real axis in the p-plane since

$$\alpha_2(\bar{p}) = -\overline{\alpha_1(p)},\ \alpha_3(\bar{p}) = -\overline{\alpha_4(p)} \tag{2.10}$$

When $\beta = 0$, there are no crossing points and the integration is carried out along the contour $\Gamma_{10} = l_1^+ \cup l_1^- \cup l_2^+ \cup l_2^-$. (If the function $Q_n(p)$ is single-valued, the integrals I_{n1}, I_{n4} on $l_2^+ \cup l_2^-$ and the integrals I_{n2}, I_{n3} on $l_1^+ \cup l_1^-$ vanish since the corresponding integrands do not change their values on passing to the opposite edge of the cut). We note that the contour Γ_{10} cannot be used as the pass contour since, when $\beta \neq 0$, the kernels of the integrals I_{nk} have a non-integrable singularity at the branching points p_{14} or p_{23}.

Hence, the asymptotic forms of the integrals are determined by the contribution from the crossing points when $\beta \neq 0$ (the branching points when $\beta = 0$) plus the sum of the contributions from the poles of the functions $Q_n(p)$ which the straight line Γ_1 passes through when it is deformed into the crossing contour.

3. ***Asymptotic forms of the integrals I_{nk}.*** We will make use of the well-known formulae in /3/ in order to obtain the asymptotic expansions.

The principal term of the asymptotic forms can be represented in the form

$$I_{nk} = I_{nk}^0 + I_{nk}^1 \tag{3.1}$$

where I_{nk}^0 is the contribution from the poles of the functions $Q_n(p)$ and I_{nk}^1 is the contribution from the crossing points when $\beta \neq 0$ or from the branching points when $\beta = 0$.

Formulae for calculating I_{nk}^{0}. Let us introduce the function (taking account of inequalities (2.8) and Tables 1 and 2)

$$r_{nk}(p, \beta) = \frac{i \exp\{[-{}^1/_4\lambda + S_k(p, \beta)]\, t - i\alpha_k(\eta - \xi_n)\}}{2\Phi_{\alpha}'(\alpha_k, p)} \tag{3.2}$$

into the treatment and use the notation $D_k = \{\mathrm{Re}\, S_k(p, \beta) > 0\} \cup \{\mathrm{Re}\, p > 0\}$ $(k = 1, 2, 3, 4)$, p_j are the poles of the function $Q_n(p)$.

Then,

$$I_{nk}^{0} = \sum_j \chi_j r_{nk}(p_j, \beta) \operatorname{res} Q_n(p_j) \tag{3.3}$$

and, for $\beta = 0$ $\chi_j = 0$ when $\mathrm{Re}\, p_j < 0$ (small terms of the order of $O(e^{-ct})$, $c > 0$), were discarded in calculating the asymptotic forms of the integrals I_{nk}), $\chi_j = 2$ when $\mathrm{Re}\, p_j \geqslant 0$ and, for $\beta \neq 0$ $\chi_j = 0$ when $p_j \notin D_k \cup \Gamma_{nk}$, $\chi_j = 1$ when $p_j \in \Gamma_{nk} \setminus p_k^*$, and $\chi_j = 2$ when $p_j \in D_k$.

Formulae for calculating I_{nk}^{1}. Let just the poles be the singular points of the functions $Q_n(p)$ (this condition holds when there is no carriage). We introduce the notation

$$\theta_{1k} = 4\{2\pi\, |\omega + v\alpha_k|\, [3\alpha_k^2 - (\beta + v)^2 + {}^1/_2 k_1]\}^{1/2} \tag{3.4}$$
$$\theta_{2k} = \exp\{[-{}^1/_4\lambda + i(\omega - \beta\alpha_k)]\, t + {}^1/_4 i\varkappa_k\pi\}$$
$$\theta_{3k} = \exp[(-{}^1/_4\lambda + i\omega)\, t + {}^1/_8 i\varkappa_k\pi], \quad \varkappa_k = \begin{cases} 1, & k = 1, 4 \\ -1, & k = 2, 3 \end{cases}$$

where $\alpha_k = \alpha_k(i\omega)$ are the roots of the polynomial (2.3).

Let us consider the following cases.

1°. The poles of the functions $Q_n(p)$ do not coincide with the crossing and branching points. Then,

$$I_{nk}^{1} = \theta_{2k}\left\{i(-1)^{k+1} \frac{Q_n(i\omega)}{\sqrt{\pi}\, \theta_{1k}} \exp[-i\alpha_k(\eta - \xi_n)]\, t^{-1/2} + O(t^{-3/2})\right\} \tag{3.5}$$

where, when $\beta \neq 0$, $\beta \neq -v$ or $\beta = -v$ $(k_1 > 0)$ $\omega = \mathrm{Im}\, p_k^*$ for $k = 1, 2, 3, 4$ when $\beta = 0$ $(v > 0$ or $k_1 > 0)$ $\omega = \mathrm{Im}\, p_{14}$ for $k = 1, 4$ and $\omega = \mathrm{Im}\, p_{23}$ for $k = 2, 3$ and

$$I_{nk}^{1} = \theta_{3k}\left[i(-1)^{k+1} \frac{\Gamma({}^1/_4)}{8\sqrt{2}\, \pi} |\omega|^{-3/4} Q_n(i\omega)\, t^{-1/4} + O(t^{-3/4})\right] \tag{3.6}$$

where, when $\beta = -v$ $(v > 0,\ k_1 = 0)$ $\omega = \varkappa_k \sqrt{\gamma/2}$ for $k = 1, 2$ and, when $\beta = 0$ $(v = k_1 = 0)$ $\omega = \varkappa_k \sqrt{\gamma/2}$ for $k = 1, 2, 3, 4$.

2°. A pole of the function $Q_n(p)$ coincides with the crossing point $(\beta \neq 0)$. Let us consider the function

$$Q_n(p) = D_n(p)/(p - p_0) \quad (p_0 = p_k^*) \tag{3.7}$$

Then,

$$I_{nk}^{1} = \theta_{2k}\left\{i(-1)^{k+1} \frac{1}{\sqrt{\pi}\, \theta_{1k}} \left[D_n'(i\omega) - \frac{1}{\beta}\left(\frac{iv}{\omega + v\alpha_k} + \right.\right.\right. \tag{3.8}$$
$$\left.\left.\left. i\psi_{nk} + \eta - \xi_n\right) D_n(i\omega)\right] \exp[-i\alpha_k(\eta - \xi_n)]\, t^{-1/2} + O(t^{-3/2})\right\}$$
$$\psi_{nk} = \{9\alpha_k^4 + [3k_1 - (2v + 3\beta)(3v + \beta)]\, \alpha_k^2 - 2\beta\omega\alpha_k +$$
$${}^1/_4 k_1^2 - {}^1/_2 k_1 (v + \beta)(2v + \beta) + v(v + \beta)^3\} \times$$
$$\{\beta\, [6\alpha_k^2 + k_1 - 2(v + \beta)^2]\, (\omega + v\alpha_k)\}^{-1}$$

where $\beta \neq -v$ or $\beta = -v$ $(k_1 > 0)$, $\omega = \mathrm{Im}\, p_k^*$ for $k = 1, 2, 3, 4$ and

$$I_{nk}^{1} = \theta_{3k}\left\{i(-1)^{k+1} \frac{\Gamma({}^1/_4)}{8\sqrt{2}\, \pi} |\omega|^{-3/4} \times \right. \tag{3.9}$$
$$\left.\left[D_n'(i\omega) + \left(\frac{19i}{20\omega} + \frac{\eta - \xi_n}{v}\right) D_n(i\omega)\right] t^{-1/4} + O(t^{-3/4})\right\}$$

where, when $\beta = -v$ $(v > 0,\ k_1 = 0)$ $\omega = \varkappa_k \sqrt{\gamma/2}$ for $k = 1, 2$.

3°. A pole of the function $Q_n(p)$ coincides with a branching point ($\beta = 0$). We consider expression (3.7) when $p_0 = p_{14}$ or $p_0 = p_{23}$ and transform the initial integrals (2.8) in the following manner:

$$I_{nk} = \frac{1}{2\pi i} \exp[(-{}^1/_4\lambda + p_0)t] I^*_{nk}$$

$$I^*_{nk} = \int_{\Gamma_1} \frac{iD_n(p)\exp[-i\alpha_k(\xi - \xi_n)]}{(p - p_0)\Phi_\alpha'(\alpha_k, p)} \exp[(p - p_0)t]\, dp$$

whence

$$\frac{d}{dt} I^*_{nk} = \int_{\Gamma_1} \frac{iD_n(p)\exp[-i\alpha_k(\xi - \xi_n)]}{\Phi_\alpha'(\alpha_k, p)} \exp[(p - p_0)t]\, dp$$

In order to obtain the asymptotic forms of the last integral we make use of formulae (3.5) and (3.6) when $\beta = 0$. Then, by integrating with respect to time, we get

$$I^1_{nk} = \theta_{2k}\left[2i(-1)^{k+1}\frac{D_n(i\omega)}{\sqrt{\pi}\,\theta_{1k}} \exp[-i\alpha_k(\xi - \xi_n)]\, t^{1/2} + O(1)\right] \tag{3.10}$$

where, when $v > 0$ or $k_1 > 0$ $\omega = \operatorname{Im} p_{14}$ for $k = 1.4$ and $\omega = \operatorname{Im} p_{23}$ for $k = 2,3$.

$$I^1_{nk} = \theta_{3k}\left[i(-1)^{k+1}\frac{\Gamma(^1/_4)}{6\sqrt{2}\,\pi}|\omega|^{-3/4} D_n(i\omega)\, t^{3/4} + O(1)\right] \tag{3.11}$$

where $v = k_1 = 0$, $\omega = \varkappa_k\sqrt{\gamma/2}$ for $k = 1, 2, 3, 4$.

4°. The function $Q_n(p)$ has a pole of order $m \geqslant 1$ which coincides with a crossing or a branching point. Let us expand the function $Q_n(p)$ in a Laurent series in the neighbourhood of the crossing (branching) point. Next, by reasoning using the method which has been described above, we get

$$I_{nk}{}^1 = O(t^{m-1/2}) \quad (\beta = 0,\ v > 0 \text{ or } k_1 > 0)$$

$$I_{nk}{}^1 = O(t^{m-1/4}) \quad (\beta = 0,\ v = k_1 = 0)$$

$$I^1_{nk} = \begin{cases} O(t^{1/2 m-1/2}), & m - \text{even} \\ O(t^{1/2 m-1}), & m - \text{odd} \end{cases} \qquad (\beta \neq 0, \beta \neq -v)$$

$$I^1_{nk} = \begin{cases} O(t^{1/2 m-1/4}), & m - \text{even} \\ O(t^{1/2 m-3/4}), & m - \text{odd} \end{cases} \qquad (\beta = -v,\quad v > 0)$$

When the carriage is present, the poles and branching points p_{14}, p_{23}, at which the functions $Q_n(p)$ can have a zero value, are the singular points of the functions $Q_n(p)$. Hence, when $\beta \neq 0$, the asymptotic forms which have been presented above remain completely valid and, when $\beta = 0$, it may be possible to refine them depending on the actual construction of the carriage.

In concluding, we note that, if $Q_n(\bar{p}) = \overline{Q_n(p)}$, then, when account is taken of (2.10), the equalities

$$I_{n1}{}^1 = \overline{I_{n2}{}^1},\ I_{n4}{}^1 = \overline{I_{n3}{}^1}$$

hold.

4. ***Analysis of the asymptotic solutions for certain types of loads.*** Let us consider the case when $N = 1$, $\xi_1 = 0$ (the unit subscript is subsequently omitted).

The action of a concentrated force. Let a pulsed force

$$q(t) = \delta(t),\ Q(p) \equiv G,\ I_k{}^0 \equiv 0$$

be applied to the beam.

Formulae (3.5) and (3.6) show that the frequency $\omega_* = |\operatorname{Im} p_{14}|$ (the smallest in the spectrum of the characteristic frequencies of the beam) is the fundamental, that is, the frequency which determines the form of the oscillations of the beam with the passage of time in the neighbourhood of the point where the force is applied. Curve *1* in Fig.4 illustrates the change in this frequency as the velocity v increases according to Eq.(2.5). The parameter β has the meaning of the group velocity of the motion of the wave packet and is equal to the rate of energy propagation. It follows from this that the energy transfer within the beam occurs from the point of application of the force to infinity. Here, the flexural wave does not have a front: at the moment when the force is applied, the perturbations encompass the whole beam. Actually,

$$\beta(\alpha_k) \approx \mp 2^{3/4}\omega^{1/2}, \; \alpha_k(\varkappa_k i\omega) \approx \mp 2^{1/4}\omega^{1/2}\varkappa_k\omega \to +\infty$$

that is, the group velocity is unbounded and infinitely short waves instantaneously depart to infinity. This defect is a consequence of the fact that Eq.(1.1) is of the parabolic type.

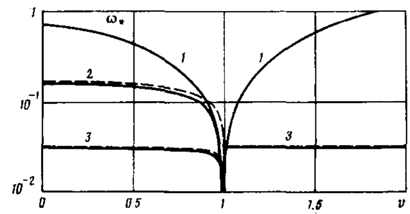

Fig. 4

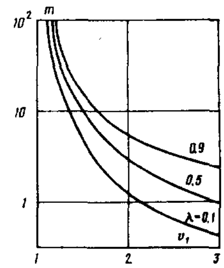

Fig.5

Let us now consider the action of a constant force

$$q(t) = H(t), \; Q(p) = (p - {}^1/_4\lambda)^{-1}$$

on the beam where $H(t)$ is the Heaviside function.

In accordance with formula (3.3), the contribution of the pole $p_1 = {}^1/_4\lambda$ describes the perturbations in the beam, the quasifront of which propagates with velocities $\beta^- < 0$ and $\beta^+ > 0$, that is,

$$\chi_1 = \begin{cases} 0, & \beta > \beta^+, \quad \beta < \beta^- \\ 1, & \beta = \beta^-, \quad \beta = \beta^+ \\ 2, & \beta^- < \beta < \beta^+ \end{cases} \quad (\lambda > 0 \text{ or } \lambda = 0, v > v_0)$$

$$\chi_1 = \begin{cases} 0, & \beta \neq 0 \\ 2, & \beta = 0 \end{cases} \quad (\lambda = 0, \; 0 \leqslant v < v_0)$$

$$\beta^- = -\frac{\lambda}{4\,\mathrm{Im}\,\alpha_1(p_1)}, \quad \beta^+ = -\frac{\lambda}{4\,\mathrm{Im}\,\alpha_4(p_1)} \quad (\lambda > 0)$$

$$\beta^{\pm} = \begin{cases} 0, & 0 \leqslant v \leqslant v_0 \\ \pm v^{-1}[(v^4 - 1) - v^2 k_1 + {}^1/_4 k_1^2]^{1/2}, & v > v_0 \end{cases} \quad (\lambda = 0)$$

where v_0 is calculated using formula (2.6).

Hence, when $v < v_0, \lambda = 0$, a symmetric bending of the beam is established with the passage of time which decreases exponentially with respect to ξ (when $\lambda > 0$, a displacement of the beam profile occurs in a direction opposite to the motion of the force), the energy of which is not radiated to infinity since there is no quasifront ($\beta^- = \beta^+ = 0$). In the course of time when $v > v_0$ the solution is represented by two since curves of differing frequency and amplitude which propagate in opposite directions with group velocities β^+ and β^-. Moreover, their quasifront is "blurred" (formulae (3.8) and (3.9)). If the force moves at the critical velocity $v = v_0$ $(\lambda = 0)$, then the I_k^1 are calculated using formulae (3.5) and (3.6) when $\beta \neq 0$ and, using formulae (3.10) and (3.11) when $\beta = 0$. It can be seen from this that a constant force brings about an unbounded increase in the bending of the beam in the neighbourhood of the point of application of the force.

The appearance of a resonance under the action of the oscillatory force

$$q(t) = H(t)\exp(i\omega_0 t), \; Q(p) = (p - i\omega_0)^{-1} \quad (\lambda = 0)$$

is especially graphic.

The coincidence of a pole $p_1 = i\omega_0$ with one of the branching points p_{14} or p_{23} (formulae (3.10) and (3.11) corresponds to the appearance of a resonance. By solving Eq.(2.5) for v, it is possible to calculate the corresponding critical velocities of the motion.

The action of a force applied to a concentrated mass. Let us now consider the action of a pulsed force $P(t) = \delta(t)$. When account is taken of Eq.(1.2), we have

$$Q(p) = [m(p - {}^1/_4\lambda)^2 h(p, 0) + 1]^{-1}$$

The branching points p_{14} and p_{23} and the two complex conjugate poles which disappear when $v = v_0$, $\lambda = 0$ serve as the singular points of the function $Q(p)$. Investigations which were carried out showed that a critical velocity of the motion of a load v_1 appears in the presence of a concentrated mass and that, when this velocity is exceeded, the poles reach the half plane $\mathrm{Re}\,p > {}^1/_4\lambda$ which leads to an exponential increase in the solution which is proportional to the time t. The dependence of m on v_1 for certain values of the viscosity λ $(k_1 = 0)$ is shown in Fig.5. Here, $v_1 \equiv v_0$ when $\lambda = 0$. Similar curves have previously been found using a different method /4/.

When $m \gg 1$, $0 \leqslant \lambda \ll 1$, $k_1 = 0$, it is possible to use an approximate formula, obtained by the perturbation method /5/, to calculate the poles of the functions $Q(p)$

$$p_{1,2} \approx {}^1/_4\lambda \pm i\,[8\,(1-v^2)]^{1/4} m^{-1/4} \quad (0 \leqslant v < 1) \tag{4.1}$$

If $v = v_0\ (\lambda = 0)$, then the function $Q(p)$ is not equal to zero at the branching points p_{14} and p_{23} and the asymptotic forms of the integrals I_k^1 are calculated using formulae (3.5) and (3.6). When $v \neq v_0\ (\beta = 0)$ the estimates

$$I_k^1 = O\,(t^{-3/2})\ (v > 0 \text{ or } k_1 > 0) \qquad I_k^1 = O\,(t^{-1/4})\ (v = k_1 = 0) \tag{4.2}$$

hold.

Hence, when $v < v_0$, $\lambda = 0$, a pulsed force gives rise to non-decaying harmonic oscillations in the neighbourhood of a concentrated mass, which are determined by the contribution of the poles p_1 and p_2 with a frequency $\omega_* = |\operatorname{Im} p_{1,2}|$ (and, correspondingly, decaying oscillations whenever there is viscosity) which is the lowest frequency in the spectrum of the characteristic frequencies of the beam.

Curves 2 in Fig.4 show the change in the frequency ω_* as a function of the velocity of the motion of a mass $m = 100$ when $k_1 = 0$ (the solid line is the exact solution and the broken line is the approximate solution obtained using formula (4.1)). We also note that the poles p_1 and p_2 do not give rise to the appearance of a quasifront when $\lambda = 0$, $0 \leqslant v < v_0$.

Let us now consider the action of an oscillating force when $\lambda = 0$

$$P\,(t) = H\,(t) \exp\,(i\omega_0 t)$$
$$Q\,(p) = (p - i\omega_0)^{-1}\,[mp^2 h\,(p,\,0) + 1]^{-1}$$

If $v = v_0$, the asymptotic forms of the integrals $I_k^1\ (\beta = 0)$ are calculated using formulae (3.5) and (3.6) when $\omega_0 \neq 0$ and (3.10) and (3.11) when $\omega_0 = 0$. If $0 \leqslant v < v_0$, then, when $\omega_0 \neq \operatorname{Im} p_{14}$ and, when $\omega_0 \neq \operatorname{Im} p_{23}$, the estimates (4.2) hold for I_k^1 while, when $\omega_0 = \operatorname{Im} p_{14}$ or $\omega_0 = \operatorname{Im} p_{23}$

$$I_k^1 = O\,(t^{-1/2})\ (v > 0 \text{ or } k_1 > 0), \qquad I_k^1 = O\,(t^{-3/4})\ (v = k_1 = 0)$$

Hence, a resonance only arises when $|\omega_0| = \omega_*$ or $\omega_0 = 0\ (v = v_0)$. In the first case, the bending of the beam increases in proportion to t (a double pole in the case of the function $Q\,(p)$) while, in the second case, it increases in proportion to $t^{1/2}$.

The action of a force applied to a spring-mounted mass. The equation of the oscillation of the mass has the form

$$m\frac{d^2z}{dt^2} - \mu\left(\frac{dy}{dt} - \frac{dz}{dt}\right) - \varepsilon\,(y - z) = P\,(t) \quad (\xi = 0)$$

Let us now consider the action of a pulsed force $P\,(t) = \delta\,(t)$:

$$Q\,(p) = a\,(p)\,b^{-1}\,(p),\ Z\,(p) = [a\,(p)\,h\,(p,\,0) + 1]\,b^{-1}\,(p)$$
$$b\,(p) = m\,(p - {}^1/_4\lambda)^2\,a\,(p)\,h\,(p,\,0) + m\,(p - {}^1/_4\lambda)^2 + a\,(p),\ a\,(p) = \mu\,(p - {}^1/_4\lambda) + \varepsilon$$
$$z\,(t) = \frac{\exp\,(-{}^1/_4\lambda t)}{2\pi i}\int_{\Gamma_1} Z\,(p)\,e^{pt}\,dp$$

The branching points p_{14} and p_{23} and the two complex conjugate poles located in the half plane $\operatorname{Re} p \leqslant {}^1/_4\lambda$ are the singular points of the functions $Q\,(p)$ and $Z\,(p)$.

When $m \gg 1$, $0 \leqslant \lambda \ll 1$, $k_1 = \mu = 0$, the approximate formulae

$$p_{1,2} \approx {}^1/_4\lambda \pm i\left\{\frac{2\varepsilon\sqrt{2\,(1-v^2)}}{m\,[2\sqrt{2\,(1-v^2)} + \varepsilon]}\right\}^{1/2} \quad (0 \leqslant v < 1) \tag{4.3}$$
$$p_{1,2} = {}^1/_4\lambda \pm i\,(\varepsilon/m)^{1/2} \quad (v = 1)$$
$$p_{1,2} \approx {}^1/_4\lambda \pm i\left\{\frac{\varepsilon\sqrt{(v^4-1)^3}}{m\,[\sqrt{(v^4-1)^3} + {}^1/_4\lambda v\varepsilon]}\right\}^{1/2} \quad (v > 1)$$

obtained by the perturbation method, can be used to calculate the poles of the function $Q(p)$.

As in the case of the motion of a concentrated mass, the asymptotic forms of the integrals I_k^1 when $\beta = 0$ (for any $v \geqslant 0$) are estimated using formulae (4.2). The relationships

$$z(t) = \sum_{j=1}^{2} \exp[(p_j - {}^1/_4\lambda)t] \operatorname{res} Z(p_j) + I_z$$

$$I_z = \exp(-{}^1/_4\lambda t)\cos(\omega t + {}^1/_4\pi)\left[\frac{\theta_{11}}{\sqrt{\pi}\, m^2 (i\omega - {}^1/_4\lambda)^4} t^{-3/2} + O(t^{-5/2})\right]$$

$$(\omega = \operatorname{Im} p_{14},\ v > 0)$$

$$I_z = \exp(-{}^1/_4\lambda t)\cos(\omega t + {}^3/_8\pi)\left[\frac{28\sqrt{2}\,\omega^{3/4}}{m^2 (i\omega - {}^1/_4\lambda)^4}\Gamma\left(\frac{3}{4}\right) t^{-7/4} + O(t^{-11/4})\right]$$

$$(\omega = \sqrt{\gamma/2},\ v = 0)$$

hold in the case of the spring deformation function.

Hence, when $v < v_0, \lambda = 0$, a pulsed force gives rise to non-decaying harmonic oscillations of the beam in the neighbourhood of the point of contact $\xi = 0$ and of the spring-mounted mass with a frequency $\omega_* = |\operatorname{Im} p_{1,2}|$ (and, correspondingly, to decaying oscillations whenever there is viscosity) which is the lowest frequency in the spectrum of the characteristic vibrations of the beam. When $v > v_0$, the poles p_1 and p_2 are located in the half plane $\operatorname{Re} p < {}^1/_4\lambda$. Hence, the amplitude of the oscillations of the beam and of the spring-mounted mass decays exponentially. As in the case of a concentrated mass, these poles do not lead to the emergence of a quasifront when $\lambda = 0$.

Curves 3 in Fig.4 show the change in the frequency ω_* as a function of the velocity of motion of a mass $m = 100$ when $k_1 = 0$ (the solid line is the exact solution and the broken line is the approximate solution obtained using formula (4.3)).

Note that, when $\lambda = \mu = 0$, the action of an oscillating force $P(t) = \exp(i\omega_* t)$ leads to the occurrence of a resonance when $v \leqslant v_0$ and to a finite (although significant) maximum in the amplitude of the oscillations when $v > v_0$.

The author thanks V.M. Aleksandrov for his supervision and interest.

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Translated by E.L.S.